Math 451: Introduction to General Topology
Lecture 20

Remark. A product $X := \prod_{i\in I} X_i$ can also be equipped with a stronger (= more open sets) top, called the **box topology**, which is generated by the sets of the form $\prod_{i\in I} U_i$, where $U_i \subseteq X_i$ is open, in other words "open boxes". This top is however often too strong, hence less useful. For example, the box top. on $\Sigma^{\mathbb{N}}$ (where Σ is discrete) is discrete since every $\{0\} \subseteq \Sigma$ is open in Σ so for each $x \in \Sigma^{\mathbb{N}}$, $\{x\} = \prod_{n\in\mathbb{N}} \{x(n)\}$ is open in the box top. While the prod. on $\Sigma^{\mathbb{N}}$ is very useful (nontrivial limits exist): it enjoys the combinatorics like a discrete space (think of trees) and it enjoys limits like $\mathbb{R}$.

Product top = ptwise convergence. Let $X := \prod_{i\in I} X_i$ equipped with the prod. top. Then a sequence $(f_n) \subseteq X$ converges in the product top $\iff$ it converges ptwise, i.e. $(f_n(i))_{n\in\mathbb{N}} \subseteq X_i$ converges $\forall i \in I$.

Proof. $\Rightarrow$. Suppose $f_n \to f \in X$ in the prod. top. Fix $i \in I$ to show that $f_n(i) \to f(i)$ as $n \to \infty$. Let $U_i \subseteq X_i$ be an open neighb. of $f(i)$. Then the 1-base cylinder $[i \mapsto U_i]$ is an open neighb. of f, so $\forall^\infty n$, $f_n \in [i \mapsto U_i]$, i.e. $f_n(i) \in U_i$, so $\forall^\infty n$ $f_n(i) \in U_i$, hence $\lim_{n\to\infty} f_n(i) = f(i)$.

$\Leftarrow$. Let $U \subseteq X$ be an open neighb. of f and by moving to a subset, we may assume U is basic open, i.e. $U = [i_1 \mapsto U_{i_1}, \dots, i_k \mapsto U_{i_k}]$. By the assumption, we have

$$(\forall^\infty n \ f_n(i_1) \in U_{i_1}) \text{ and } (\forall^\infty n \ f_n(i_2) \in U_{i_2}) \text{ and } \dots \text{ and } (\forall^\infty n \ f_n(i_k) \in U_{i_k}),$$

so because k is finite, we have $\forall^\infty n \ (f_n(i_1) \in U_{i_1} \text{ and } f_n(i_2) \in U_{i_2} \text{ and} \dots \text{and } f_n(i_k) \in U_{i_k})$, i.e. $\forall^\infty n \ f_n \in [i_1 \mapsto U_{i_1}, \dots, i_k \mapsto U_{i_k}]$. $\square$

Example. The ptwise convergence for the space $\mathbb{R}^{[0,1]}$ of functions $[0,1] \to \mathbb{R}$ is given by the product top. In particular, the convergence for the space $C([0,1])$ of continuous func. $[0,1] \to \mathbb{R}$ is still given by the prod. top. on $\mathbb{R}^{[0,1]}$.

Recalling that the prod. top. is generated by projections proj_i, $i \in I$, the following is a generalization of the pointwise convergence statement.

Prop. Let $\{Y_i\}_{i\in I}$ be top. spaces and $f_i : X \to Y_i$. Equip X with the top. generated by $\{f_i\}_{i\in I}$. Then:

(a) Every $(x_n) \subseteq X$ converges in X $\iff$ $(f_i(x_n))_{n\in\mathbb{N}}$ converges in Y_i for all $i \in I$.

(b) Let Z be a top space and $g : Z \to X$. Then g is continuous $\iff$ $f_i \circ g : Z \to Y_i$ is continuous for each $i \in I$.

Proof. HW

Cor (instance of (b)). Let $X = \prod_{i\in I} X_i$ be equipped with the prod. top and Z be a top. space. Then a function $g : Z \to X$ is continuous $\iff$ $\mathrm{proj}_i \circ g : Z \to X_i$ is continuous for each $i \in I$.

Permanence properties of products.

Theorem.

(a) An arbitrary product of Hausdorff spaces is Hausdorff.

(b) A arbitrary product of regular spaces is regular.

(c) A ctbl product of 1^{st} (resp. 2^{nd}) ctbl spaces is 1^{st} (resp. 2^{nd}) ctbl.

(d) A ctbl product of separable spaces is separable.

(e) A ctbl product of metrizable spaces is metrizable.

Proof. (a), (b), and 1^{st} ctbility in (c) are HW

(c) 2^{nd} ctbl. Let $X := \prod_{n\in\mathbb{N}} X_n$ and $\mathcal{B}_n$ be a ctbl basis for X_n. Let $\mathcal{B}$ be the set of finite-base cylinders $[n_1 \mapsto B_{n_1}, n_2 \mapsto B_{n_2}, \dots, n_k \mapsto B_{n_k}]$, where $k \in \mathbb{N}^+$, $B_{n_i} \in \mathcal{B}_{n_i}$. Then $\mathcal{B}$ is ctbl because there are only ctbly-many finite subsets of $\mathbb{N}$ ($|\mathcal{P}_{fin}(\mathbb{N})| \le \mathbb{N}^{<\mathbb{N}}$). To check that $\mathcal{B}$ is a basis, it suffices to show that every finite-base cylinder $[n_1 \mapsto U_{n_1}, \dots, n_k \mapsto U_{n_k}]$ is a union of sets in $\mathcal{B}$. But each U_{n_i} is a union $\bigcup_{\ell\in\mathbb{N}} B_{n_i}^{(\ell)}$ of sets from $\mathcal{B}_{n_i}$, so $[n_1 \mapsto U_{n_1}, \dots, n_k \mapsto U_{n_k}] = \bigcup_{\ell_1\in\mathbb{N}} \bigcup_{\ell_2\in\mathbb{N}} \dots \bigcup_{\ell_k\in\mathbb{N}} [n_1 \mapsto B_{n_1}^{(\ell_1)}, \dots, n_k \mapsto B_{n_k}^{(\ell_k)}]$. $\square$

(d) Let $X := \prod_{n\in\mathbb{N}} X_n$, where $X_n \neq \emptyset$ and let $D_n \subseteq X_n$ be ctbl dense in X_n.

Let $f_0 \in X$, i.e. $f_0(n) \in X_n$ for each $n \in \mathbb{N}$ (uses AC). Let $\mathcal{D} \subseteq X$ be the set of all points $f \in X$ which are equal to f_0 on all but fin-many coordinates, and on those finitely many coordinates it is from the respective D_n, i.e.

$$\mathcal{D} := \{f \in X : \exists N \ \forall n \geq N \ f(n) = f_0(n) \text{ and } \forall n < N \ f(n) \in D_n\}.$$

Then $\mathcal{D}$ is ctbl because $\mathcal{D} = \bigcup_{N \in \mathbb{N}} \{f \in X : \forall n \geq N \ f(n) = f_0(n) \text{ and } f|_N \in D_0 \times D_1 \times \dots \times D_{N-1}\}$ and ctbl union of ctbl sets is ctbl, as well as each $D_0 \times \dots \times D_{N-1}$ is ctbl.
To show the density of $\mathcal{D}$, it is enough to verify that $\mathcal{D}$ meets every cylinder $[0 \mapsto U_0, 1 \mapsto U_1, \dots, k \mapsto U_k]$. But for each $i \leq k$, $\exists \, d_i \in D_i \cap U_i$, so the element $f := (d_0, d_1, \dots, d_k, f_0(k+1), f_0(k+2), f_0(k+3), \dots) \in \mathcal{D} \cap [0 \mapsto U_0, \dots, k \mapsto U_k]$. □

(e) Let $X = \prod_{n \in \mathbb{N}} X_n$ and let d_n be a compatible metric on X_n. Then the metric $d_n' := \min(1, d_n)$ is still a metric on X_n defining the same top since every d_n-ball is a union of d_n-balls of radius < 1, which are also d_n'-balls. Thus, WLOG, may assume $d_n \leq 1$ to begin with. Define $d : X \times X \to [0,1]$ by setting for $f, g \in X$,

$$d(f,g) := \sum_{n \in \mathbb{N}} 2^{-(n+1)} \cdot d_n(f(n), g(n)).$$

To verify that product-open sets are d-open, it suffices to show that the cylinders of the form $[n \mapsto U_n]$, where $U_n \subseteq X_n$ is an open ball, are d-open, i.e. are unions of d-open balls in X. This is not hard to check (HW).
To verify that d-open sets are product open, use the following general claim:

Claim. It suffices to show that for every d-open ball $B_r^d(f)$, $r > 0$ and $f \in X$, $\exists$ product open $V \subseteq X$ with $f \in V \subseteq B_r^d(f)$.

Proof. Suppose this property holds and let $U \subseteq X$ be a d-open set in X. Thus, for each $f \in U$, $\exists \, B_{r_f}^d(f) \subseteq U$ for some $r_f > 0$. Then $\exists \, V_f \subseteq X$ product-open s.t. $f \in V_f \subseteq B_{r_f}(f)$. But then $U = \bigcup_{f \in U} V_f$. □

From this claim, the proof is an exercise (HW). □